Integrable and Nonintegrable Structures in Shipov's Physical Vacuum.

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The structures of a space of absolute parallelism are investigated in terms of differential topology and the possible additional constraints imposed by equivalence classes of metrics. A connection is made between projective spaces of absolute parallelism and those physical systems that can be described by a 1-form of Action.

Preface

This article started out to be a more or less detailed (but partial) commentary on G .I. Shipov's book "The Theory of the Physical Vacuum" (Evidently printed by the Moscow ST-Center, Russia, 1998 ISBN 5-7273-0011-8). The book arrived from the US on 08/17/99 at my mas in France, thanks to V. Poponin). Shipov had a email address of shipov@aha.ru (but the book was written evidently when he was a member of the International Institute for Theoretical and Applied Physics, RANS) and V. Poponin had an email address of vpo-ponin@sj.znet.com (summer 1999). I do not know if the two email addresses are valid at present. I got the book by sending $115.00 to V. Poponin, who mailed a copy to me in France. Poponin's address was

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As I studied and reread Shipov's work in more detail it became evident that there was need for clarification in certain areas. I got confused in the detail of tensor notation caused by certain ambiguities of index conventions. That which follows is an attempt to make (at least part of) Shipov's work more transparent by taking care of notation and producing both a matrix and an index presentation. The order of matrix products is important.

Introduction

In classical mechanics, the vacuum is typically characterized to be an empty void with a 3D euclidean structure, and euclidean substructures. Particles are considered as entities placed in this void which undergo motions characterized by a parameter, called time. The theory of electromagnetic fields extends the mechanical concept of the vacuum to a 4D variety that will support electromagnetic waves. In order to preserve the singular solutions (propagating electromagnetic discontinuities defined as signals [1]) to Maxwell's theory, the 4D vacuum apparently must have some structure. That is, the singular solutions belong to an equivalence class of 4D systems related to one another either by the linear Lorentz transformation, or by the non-linear Mobius (projective) transformations [2]. If different observers are related by a Lorentz transformation to the vacuum, then if one observer detects a propagating electromagnetic discontinuity, so will the Lorentz related observer detect a propagating electromagnetic discontinuity. The vacuum is then defined as the restricted 4D variety that is constrained by the imposition of the Minkowski metric, and the varieties equivalent to it under the transformations mentioned above. In practice, the non-linear Mobius transformations are mostly ignored.

The metric theory of gravitational fields seems to indicate that the vacuum with a gravitational field must have metric structure that is not among the equivalence class of Lorentz transformations that include the Minkowski pseudo euclidean metric. Quantum theory further complicates the view of the vacuum by consideration of virtual states and pair production. The question remains: Just what is the physical vacuum? It apparently is not the void of a euclidean space with euclidean substructures, but current science seems to indicate that it is at least a 4D variety.

In his book ”The Theory of the Physical Vacuum” G. Shipov offers the interesting conjecture that the Physical Vacuum is 4 dimensional space of Absolute Parallelism. As a 4 dimensional euclidean space is a space of absolute parallelism, at first glance it would appear that the Shipov definition of the vacuum is far too constrained to be of interest. Although euclidean spaces are examples of (often said to be ”trivial”) spaces of absolute parallelism, but spaces of absolute parallelism are not just euclidean ”voids”. Spaces of absolute paralleelism can admit definite substructures that (according to Shipov) have physical significance. For this idea Shipov deserves applause.

Consider a 4D variety that supports a matrix of C2 functions such that over the domain of support the determinant of the matrix is never zero. The matrix is defined as the Basis Frame, or Repere Mobile. The differentiable basis frame permits the decomposition of vector fields not only at every point, but also in their nearby neighborhoods through a well defined linear connection. As the basis frame $[F]$ never has a zero determinant, the inverse basis frame $[G]$ exists globally and is well defined algebraically. A linear connection is derived by exterior differentiation of the identity $[F]\circ[G]= [I]$ to give either

$$d[F]-[F][C]=0 \quad or \quad d[F]+[\Delta][F]=0.$$

The matrix $[C]$ is defined as the right Cartan matrix of connection 1-forms. The matrix $[\Delta]$ is defined as the left Cartan matrix of connection 1-forms. These connection equations are used to define what is meant by parallel transport. [3]

By repeated exterior differentiation it is possible to show that the Cartan matrix of curvature 2-forms $[\Theta]$ vanishes:

$$[\Theta] \triangleq \{[C]\hat{}[C]+d[C]\} \Rightarrow [0]$$

Next, consider the representation of a set of differentials in terms of the basis frame:

$$|dx^k\rangle = [I]\circ|dx^k\rangle = [F]\circ[G]\circ|dx^k\rangle = [F]\circ|\sigma\rangle$$

Repeated exterior differentiation implies that the Cartan vector of Torsion 2-forms, $|\Sigma\rangle$ vanishes:

$$|\Sigma\rangle \triangleq \{[C]\circ|\sigma\rangle + |d\sigma\rangle\} \Rightarrow 0.$$

A space of absolute parallelism is defined as a variety for which the both Cartan vector of torsion 2-forms $|\Sigma\rangle$ and the Cartan matrix of curvature 2-forms $[\Theta]$ vanish. The development above thereby shows that any space that supports a global basis frame with non-zero determinant is a domain of Absolute Parallelism. Perhaps the simplest example is a projective space which supports at each point a projective matrix of functions with non-zero determinant. These matrices define the general linear group, which has two components. The first component admits an identity element (det >1), but the second component does not (det <1).

From this point forward the focus will be on such varieties that support a global Basis Frame of functions. Such spaces of absolute parallelism can have subspaces of interesting and unexpected structure. For example, spaces of absolute parallelism can support a right Cartan matrix of connection one forms that have certain anti-symmetry properties (of the 3 indexed coefficients C^a_{bc}) associated with the torsion of an affine connection ($C^a_{bc} - C^a_{cb} \neq 0$). Yet the Cartan vector of torsion 2-forms is zero! (This observation emphasizes that the two types of torsion are not the same.) If the Frame matrix can be constructed from the Jacobian matrix of a parametric integrable map, then the Cartan connection matrix is always symmetric (in the lower indices), and the concept of Affine torsion is empty ($C^a_{bc} - C^a_{cb} = 0$). Exact integrable maps imply that the domain is free of Affine torsion. Moreover, if the Jacobian matrix of the map is orthogonal, then the three index connection coefficients of the right Cartan matrix are precisely the same as those constructed from the Christoffel formula based on an induced metric.

From that which was mentioned above, the question arises: Is the arbitrary basis frame $[F]$ imposed on a variety uniquely integrable such that the parametric map can be determined? Note that the concept of a space of absolute parallelism does not require such unique integrability. In fact, if the basis frame is uniquely integrable, the connection is affine torsion free. The question of unique integrability leads to the concept of "topological" torsion, which is not the same as the affine torsion, nor the same as Cartan's torsion 2-forms. Topological torsion is related the arbitrary Frobenius integrability of the Basis Frame, or Repere Mobile, and its inverse, which are characteristic of a space of absolute parallelism.

The first question of integrability is: Given $[F^k_a(\xi^b)]$ as a basis frame, are the linear combinations $\delta^k = F^k_a(\xi^b)d\xi^a$ perfect differentials, $\delta^k \Rightarrow dx^k$? A necessary condition is that the exterior derivative of δ^k must vanish. The requirement can be expressed by the equation.

$$\boxed{exact\ \ integrability : d([F^k_a(\xi^b)]\hat{}|d\xi^a\rangle) = [F^k_a(\xi^b)] \circ [C]\hat{}|d\xi^a\rangle \Rightarrow 0.}$$

The equation requires that $[C]\hat{}|d\xi^a\rangle \Rightarrow 0,$ which implies that the right Cartan matrix must consist of coefficients such that $(C^a_{bc} - C^a_{cb} = 0).$ Hence exact integrability of the basis frame precludes affine torsion.

However, it may be true that the δ^k admit integrating factors $\lambda^k(\xi^a).$ That is, each row of the given Frame matrix can be multiplied by a function to yield a new Frame matrix $[F]'$ such that the new $\delta^{k\prime}$ are exact.

$$\boxed{[F]' = [diagonal\ \lambda^k(\xi^a)] \circ [F]}$$

The affine torsion relative to the old frame need not be zero, and yet through integrating factors, the original non-integrable to exactness Frame can be transformed (globally) to an integrable one. For this to be possible it must be true that the topological torsion of each δ^k must vanish:

$$\boxed{Frobenius\ \ integrability\ \ requires : \delta^k\hat{}d(\delta^k) = 0.}$$

Such integrable, but not exact, frames can support affine torsion such that $(C^a_{bc} - C^a_{cb} \neq 0).$ This result has its image in hydrodynamics. Potential flows are streamline flows without vorticity ($curl v = 0$). There exists a parametric map to define the flow, and the flow is free of affine torsion. Lamb flows are streamline flows where the vorticity is not zero. However, the velocity field components are such that they satisfy the Frobenius condition. Hence there exists a parametric map, but the original flow admits affine torsion.

It may be true that a particular Frame field admits integrability for none of its δ^k, part of its δ^k, or for all of its δ^k. For example the Hopf Map generates a Frame which has 3 integrable δ^k and 1 non-integrable δ^k. On the other hand, the instanton frame has 3 non-integrable δ^k and 1 integrable $\delta^k.$

This idea of spaces of absolute parallelism was also of interest to me, as I had (over the years) managed to understand Cartan's structural equations, which on first reading of Cartan seem to appear "out the blue". By use of (Whitney) embedding techniques, I could formulate examples of spaces involving non-zero Cartan curvature 2-forms and non-zero Cartan torsion 2-forms as subspaces of Euclidean spaces. The subspaces of euclidean spaces can be used to define most properties of a manifold. However, for spaces of absolute parallelism, the algebra of Cartan's structural equations is nicely exhibited in the guise of projective geometries of dimension n+1.

See http://www22.pair.com/csdc/pdf/defects2.pdf

Projective geometries of absolute parallelism

Recall that a projective geometry is represented by a n+1 dimensional domain that supports an n+1 by n+1 Projective Matrix of functions with arguments over the n+1 independent variables. The sole requirement of such a "Projective Matrix" is that the matrix determinant is not equal to zero, thereby defining the projective domain of support of the independent variables. Such projective spaces are spaces of absolute parallelism, for the Cartan structural equations (as will be shown below) vanish over those values of independent variables which form the projective domain of support. That is, the Cartan curvature 2-forms and the Cartan torsion 2-forms vanish for projective geometries.

Projective geometries can be further constrained to form equivalence classes of Projective Matrices. For example, a subgroup of the projective group of matrices are those with determinant +1. Such projective transformations preserve orientation. Another example of a subgroup defines the class of particle Affine transformations. These are examples of subgroups that do not require a metric constraint. Further constraints can be imposed by specification of a metric.

On the domain of support, a Projective Matrix may be viewed as a matrix of linearly independent basis vectors which can be used to define arbitrary vector quantities on the domain. An equivalence class of basis matrices, has an element that can be used to define a Basis Frame $[F]$, or as Cartan puts it , "the Repere Mobile." A major question is how a given class of Basis Frames are differentially constrained in their neighborhoods. This question leads to the idea of a Connection. There are two types of connections, depending on whether an active or passive viewpoint is utilized. Herein the two points of view will be distinguished by the definitions given below for the "Cartan (left) Connection" and the "Cartan (right) Connection". It is a remarkable feature that both methods are applicable.

Due to notational inconsistencies (from this reader's point of view), it is not clear whether Shipov favors the left or the right Cartan connection. On p.181, and assuming that the symbol $\mathbf{e}^i_b$ stands for an array of contravariant columns, it would appear that Δ^i_{jk} as defined by eq (0.2) on p.181 and (5.18) on p. 187 is associated with the left Cartan connection. However on page 185 Shipov states that $\mathbf{e}^b_j$ transforms as a (contra) vector, but then gives the rule for co-vector transformation in eq (5.3). This is very confusing. In that which follows it will be presumed that Shipov focuses most attention on Δ^i_{jk}, the Cartan (left) connection. In order to distinguish this point of view, hereafter the Cartan (left) connection will be called the Shipov connection, and the Cartan (right) connection will be called the Cartan connection. Certain anti-symmetric portions of the Shipov connection will be defined as the Shipov torsion, where certain anti-symmetric portions of the Cartan connection will be defined as Affine torsion. The two concepts are not the same, for the anti-symmetries are different. For example, the anti-symmetries of the Cartan connection are zero for basis frames which are Jacobian matrices of parametric mappings. (Affine torsion does not exist for integrable parametric maps.) The anti-symmetries of the Shipov connection (defined as Shipov torsion) are not zero for the same types of basis frames.

It should be emphasized before going any further that the concept of the Cartan vector of Torsion 2-forms (which vanish on a space of absolute parallelism) is not equivalent to the concept of either Shipov torsion or Affine torsion, both of which need not vanish on a space of absolute parallelism.

Although most interest will be placed upon the domain of support, the regions for which the determinant of the matrix of functions vanishes (the compliment to the domain of support) determines hypersurfaces with specific and important topological features associated with its compliment. These hypersurfaces sometimes can be interpreted as non-stationary domains of propagating discontinuities, which are useful in defining shock waves, shear wakes, and electromagnetic signals in physical systems.

The Shipov Connection vs. the Cartan Connection

The translation from Russian to English in Shipov's book is very satisfactory (there are only a few places where the English words used in the translation seem strange and do not flow smoothly), but there are a significant number of typos, which in a book such as this one, which makes heavy duty extensive use of tensor indices, can lead to difficulties now and then. The major problem for me was one of notation, and as is usual in many tensor treatments, what are the independent variables. For example Shipov on page 185 defines $\mathbf{e}_j^b$ as a (contra-variant) vector and the symbol $\mathbf{e}_b^j$ as a co vector, but this is inconsistent with the transformation rule given by eq. (5.3) - which is the classic rule for covariants, and classic tensor conventions where the upper index is the contra index and the lower index is the co index. The more I study Shipov's book, the more confused I get in deciding which (in Shipov's mind) is the basis frame (of contravariant column vectors) and which is the inverse basis frame. Therefor I have rewritten the original commentary on Shipov's book into a more well defined format. I hope that I have not mis-interpreted Shipov's concepts.

I am used to the constraint (for a space of absolute parallelism) that the exterior differential of a basis frame be closed, forming a differential ideal. This closure constraint is a topological idea. That is, the exterior differential of any basis vector in the set is a linear combination of basis vectors of the set. This concept of closure is also in the spirit of the Cartan use of exterior algebra: the exterior product of elements of the algebra are closed in a finite sense.

From the definition of a Basis Frame matrix of functions, $[F]$, its inverse, $[G]$, and the equation

$$[F] \circ [G] = [1],$$

it is remarkable that the differentials of the basis frame, $d[F]$, can be expressed in two different ways. The closure point of view places emphasis on the Cartan connection, $[C]$ a matrix of 1-forms defined such that

$$\text{Cartan} \;:\; d[F] - [F] \circ [C] = 0.$$

The (right) Cartan connection $[C]$ acts on the basis frame $[F]$ from the right. On the other hand Shipov seems to place emphasis on a different point of view based on the formula,

$$Shipov : \;\; d[F] + [\Delta] \circ [F] = 0.$$

The left Cartan or Shipov connection $[\Delta]$ acts on the basis frame $[F]$ from the left.

The two formulations are not the same, for the Cartan (right matrix) of connection 1-forms implies that the differential of any column basis vector is a linear combination of all column basis vectors of the set.

$$d\left|\begin{matrix} e_a^1 \\ e_a^2 \\ e_a^3 \\ e_a^4 \end{matrix}\right\rangle = \left|\begin{matrix} e_1^1 \\ e_1^2 \\ e_1^3 \\ e_1^4 \end{matrix}\right\rangle C_a^1 + \left|\begin{matrix} e_2^1 \\ e_2^2 \\ e_2^3 \\ e_2^4 \end{matrix}\right\rangle C_a^2 + \left|\begin{matrix} e_3^1 \\ e_3^2 \\ e_3^3 \\ e_3^4 \end{matrix}\right\rangle C_a^3 + \left|\begin{matrix} e_4^1 \\ e_4^2 \\ e_4^3 \\ e_4^4 \end{matrix}\right\rangle C_a^4$$

This is a concept of closure, a concept which is inherent in much of Cartan's work. The formulation is related to a "passive" interpretation of the action of the total differential on any basis column vector.

The Shipov Connection (or Cartan left matrix connection) operates on the components of a given vector

$$d\left|\begin{matrix} e_a^1 \\ e_a^2 \\ e_a^3 \\ e_a^4 \end{matrix}\right\rangle = \left|\begin{matrix} \Delta_1^1 e_a^1 + \Delta_2^1 e_a^2 + \Delta_3^1 e_a^3 + \Delta_4^1 e_a^4 \\ \Delta_1^2 e_a^1 + \Delta_2^2 e_a^2 + \Delta_3^2 e_a^3 + \Delta_4^2 e_a^4 \\ \Delta_1^3 e_a^1 + \Delta_2^3 e_a^2 + \Delta_3^3 e_a^3 + \Delta_4^3 e_a^4 \\ \Delta_1^4 e_a^1 + \Delta_2^4 e_a^2 + \Delta_3^4 e_a^3 + \Delta_4^4 e_a^4 \end{matrix}\right\rangle.$$

The Shipov connection implies that the differential of any column vector is an active operation on that column vector, and does not involve the other column basis vectors.

The remarkable feature of spaces of absolute parallelism is that they offer these two points of view. One involves the change of a reference frame and the other involves an operation in a fixed frame. This duality apparently extends the idea of relativity beyond that of a metrically constrained system.

It will be demonstrated below that the two connection matrices of 1-forms $[C]$ and $[\Delta]$ are conjugates related by a (negative) similarity transformation (where $[F] \circ [G] = [1]$),

$$[\Delta] = -[F] \circ [C] \circ [G].$$

Hence to a factor of minus 1, the similarity invariants of both connections are the same. From a topological point of view, this implies that different spaces of absolute parallelism can be classified according to their similarity invariants. This topic is not discussed by Shipov, but will be taken up by the present author.

The Integrable Parametric Case

The (right) Cartan connection

In order to set the stage, to get rid of notational inconsistencies, and to make the understanding of spaces of absolute parallelism a bit more transparent, consider the special case of a space of absolute parallelism defined by a parametric map, ϕ, from n variables (or parameters) $\{\xi^b\}$ of the initial state into a space of n variables $\{x^k\}$ of the final state.

$$\phi : \xi^b \Rightarrow x^k = \phi^k(\xi^b)$$

An example is given by the position vector with components $\{ct, x, y, z\}$ given in terms of spherical coordinates $\{ct, rsin(\theta)cos(\varphi), rsin(\theta)sin(\varphi), rcos(\theta)\}$. The differentials are related by the equation

$$d\phi : |d\xi^a\rangle \Rightarrow |dx^k\rangle = [\partial\phi^k(\xi^b)/\partial\xi^a] \circ |d\xi^a\rangle$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \sin(\theta)\cos(\varphi) & r\cos(\theta)\cos(\varphi) & -r\sin(\theta)\sin(\varphi) \\ 0 & \sin(\theta)\sin(\varphi) & r\cos(\theta)\sin(\varphi) & r\sin(\theta)\cos(\varphi) \\ 0 & \cos(\theta) & -r\sin(\theta) & 0 \end{bmatrix} \circ \left| \begin{matrix} d(ct) \\ dr \\ d\theta \\ d\varphi \end{matrix} \right\rangle$$

The matrix of partial differentials is the Jacobian matrix of functions with arguments on the initial state. No metric and no domain of support has been specified. In that which follows the domain of support is defined as that set of values ξ^b on the initial state, where the Jacobian determinant does not vanish $(r^2 \sin(\theta) \neq 0)$. The Jacobian matrix can be viewed as a matrix of contravariant vectors (on the final state, x^k) in columns, and can be used as a basis frame (with arguments on the initial state ξ^b) on the domain of support (where det$[\partial\phi^k(\xi^b)/\partial\xi^a] \neq 0$). That is, assume the basis frame is given by a set of contravariant columns with row index k and column index a and with arguments on ξ^b :

$$[F^k_a] = [\partial\phi^k(\xi^b)/\partial\xi^a].$$

As yet there has been no metric imposed upon the space, but even without specification of a metric it is possible to use the general formulas given above to define a Cartan connection

$$d[F] = [F] \circ [C] = [F] \circ [-d[G] \circ [F]] = [F] \circ [[G] \circ d[F]]$$

or a Shipov connection,

$$d[F] = [\Delta] \circ [F] = [-[F] \circ d[G]] \circ [F] = [d[F] \circ [G]] \circ [F]$$

In classical tensor analysis, the concept of an affine connection is associated with the (right) Cartan matrix (ref. L. Brand) as follows: (Remember, all the functions have arguments ξ^c)

$$d[F^k_a] = [F^k_b]\circ[C^b_{ac}d\xi^c] = [F^k_b]\circ\left[\left[G^b_j\right]\circ d\left[F^j_a\right]\right]$$
$$= [F^k_b]\circ\left[\left[G^b_j\right]\circ[\{\partial^2\phi^j(\xi^m)/\partial\xi^c\partial\xi^a\}d\xi^c]\right].$$

As the system is integrable and (assumed) twice differentiable, it follows that the coefficient functions of the connection are symmetric

$$C^b_{ac} \Rightarrow \Gamma^b_{ac} = \Gamma^b_{ca} = \left[G^b_j\right]\circ[\{\partial^2\phi^j(\xi^m)/\partial\xi^c\partial\xi^a\}d\xi^c].$$

In fact, if one computes the pullback metric g_{ab} on the initial domain $\{\xi^c\}$ induced by the quadratic form on the final state, $\eta_{jk}dx^j dx^k$

$$[g_{ab}(\xi^c)] = \left[F^j_a\right]\circ[\eta_{jk}]\circ[F^k_b],$$

and then uses the classic Christoffel formulas for deriving a connection from a metric,

$$Christoffel : \Gamma^b_{ac}(\xi^c) \Rightarrow \left\{{b \atop ac}\right\} = g^{be}\{\partial g_{ce}/\partial\xi^a + \partial g_{ea}/\partial\xi^c - \partial g_{ac}/\partial\xi^e\}.$$

it follows that, for a Jacobian basis frame, the Cartan connection is the same as the Christoffel connection, and the connection is (affine) torsion free:

$$If\ [F^k_a] = [\partial\phi^k(\xi^b)/\partial\xi^a],\ then\ C^b_{ac} = \left\{{b \atop ac}\right\}.$$

In general, the basis frame $\left[F^j_a\right]$ need not be deduced from an integrable mapping. The coefficient functions of the (right) Cartan matrix, symbolized by $C^b_{ac}(\xi^e)$, are not necessarily symmetric in the lower indices, and the symmetric parts are not necessarily generated by the Christoffel formulas. It is of some interest to decompose the Cartan connection into its symmetric and anti-symmetric parts.

$$C^b_{ac} = \Gamma^b_{ac} + \omega^b_{ac}$$

It is the anti-symmetric parts, $\omega^b_{ac} = -\omega^b_{ca}$, of the (right) Cartan connection that lead to the concept of Affine torsion. Note that this has nothing to do with metric. (Compare Shipov (5.77)). It is also possible to think of the symmetric part of the Cartan connection to be composed of a Christoffel part and a non-Christoffel part.

$$\Gamma^b_{ac} = \left\{{b \atop ac}\right\} + T^b_{(ac)}$$

The combination

$$T^b_{ac} = T^b_{(ac)} + \omega^b_{ac}$$

forms what Shipov defines as the Ricci rotation coefficients.

For the integrable case, the Jacobian formulation of a basis frame does NOT generate affine torsion, and the Ricci rotation coefficients vanish. A two surface immersed into 3 euclidean dimensions does not have affine torsion. (It must be remembered that the basis frame and the connection are defined in terms of the variables of the initial state, $\{\xi^b\}$).

A Maple work sheet giving examples of these constructions can be obtained at
http://www22.pair.com/csdc/maple/a4sphjac.mws
A pdf file of the output of the Maple computation is at
http://www22.pair.com/csdc/pdf/a4sphjac.pdf

The (left) Cartan - Shipov connection.

It is assumed that Cartan left connection is the Shipov connection $[\Delta]$ defined by the equations,

$$d[F^k_b] + [\Delta^k_{ie}d\xi^e]\circ[F^i_b] = 0$$

such that

$$[\Delta^k_{ie}d\xi^e] = [F^k_a]\circ d[G^a_i] = -d[F^k_a]\circ[G^a_i]$$

For the Jacobian Frame,

$$\Delta^k_{ie} = -\partial^2\phi^k/\partial\xi^e\partial\xi^a \circ G^a_i$$

from which it is to be noted that Δ^k_{ie} is not necessarily symmetric in the ie indices. For the spherical coordinate mapping with the basis frame represented by the Jacobian matrix, there are 19 non zero partial derivative components to the Shipov connection. The anti-symmetric part of Δ^k_{ie} is defined as Shipov torsion, (5.28)

$$\Omega^k_{ie} = -1/2(\Delta^k_{ie} - \Delta^k_{ei}).$$

There are 16 non-zero components to the Shipov torsion as defined above, while the affine torsion ω^b_{ac} is zero for the same example. Shipov torsion of the left Cartan matrix is not the same as Affine torsion constructed from the anti-symmetric components of the right Cartan connection, nor is it directly related to the Cartan torsion 2-forms.

Now is where I become truly confused with notation, for Shipov not only decomposes Δ^k_{ie} into the sum of a symmetric and an anti-symmetric parts (which is reasonable), but then also decomposes the Shipov connection as

$$\Delta^k_{ie} = \{^{\,k}_{ie}\} + T^k_{ie}.$$

From the point of view of the integrable mapping given above, the Christoffel symbols are generated from the metric on the initial state (indices abc), not the final state (indices ijk). The metric on the final state are constants, hence the associated Christoffel symbols should vanish.

The details of the computations for Jacobian basis frames is given in the Maple program http://www22.pair.com/csdc/maple/a4sphjac.mws

The inverse map and the Cartan connection

Now reverse the direction of the mapping and assume that the final state is the set of variables $\{\xi^e\}$ and the initial state is defined by the variables $\{x^k\}$. The given map is of the form

$$\psi : x^k \Rightarrow \xi^b = \psi^b(x^k)$$

The differentials are related by the equation

$$d\psi : |x^k\rangle \Rightarrow |d\xi^b\rangle = [\partial\psi^b(x^k)/\partial x^k] \circ |dx^k\rangle$$

The Jacobian matrix is a set of contravariant columns on the final state, with arguments on the initial state. The basis frame is now chosen as $[F^b_k] = [\partial\psi^b(x^k)/\partial x^k]$. From the definition of the inverse

$$[F^b_k(x)] \circ \left[G^j_b(x) \right] = [1]$$

it is again possible to define a right and a left Cartan matrix. Everything is more or less the same as before, except that the abc indices become ijk indices and the arguments of the functions and differentials are x^k instead of ξ^b.

There is one important difference. The metric on the initial state is presumed to be a set of constants, $[\eta_{jk}]$, and therefore its Christoffel symbols vanish. Therefor the right Cartan matrix, now written as $C^i_{jk}(x)$ is not the same as the Christoffel symbols in the integrable case. It is possible to induce a metric inverse on the final state with arguments on the initial state by means of the construction $g^{ab}(x) = F^a_j \eta^{jk} F^b_k$, find its inverse, and then compute the inverse $g_{ab}(x)$ on the final state. Then Christoffel symbols are awkward to compute, for the map giving the x^k as functions of ξ^b has not been specified.

The Basis Frame

Shipov evidently assumes a variety of independent variables upon which can be constructed a basis frame of functions, $\mathbf{e}^i_a$ with (rmk interpretation) $a = 0,1,2,3$ as a column (lower) index and $i = 0,1,2,3$ acting as a row (upper) index. The columns of the Basis Frame may be interpreted as contravariant vectors, or alternatively the rows of the Basis Frame may be viewed as components of covariant vectors; sometimes both interpretations are applicable.. The domain of interest is constrained such that the determinant of Basis Frame, $\mathbf{e}^i_a$, consisting of independent rows or columns of functions of base variables, is not zero. It follows that an inverse matrix exists, which Shipov designates with the symbol, $\mathbf{e}^a_i$. I prefer the notation $[F^i_a]$ for the basis frame and $[G^b_j]$ for its inverse, such that the transpose and the inverse are easily distinguished, and the order of matrix products becomes obvious.

Now the matrix of functions $[F^i_a]$ used to define the Basis Frame for a space of absolute parallelism need not be generated by a parametric mapping. All that is required for the space of absolute parallelism is that the identity

$$[F^i_a] \circ [G^a_j] = [\delta^i_j] = [\mathbf{I}]$$

is valid. From this equation, the total differentials with respect to the independent variables (what ever they may be) are given by the equation,

$$d[F^i_a] \circ [G^a_j] + [F^i_a] \circ d[G^a_j] = [0].$$

(I believe that this simple formulation was first brought to my attention via the works of Coxeter). Multiplication from the right by the matrix $[F^j_b]$ produces the result

$$d[F^i_b] + [F^i_a] \circ d[G^a_j] \circ [F^j_b] = [0],$$

There are now two possibilities.

$$Possibility\ 1 : d[F^i_b] + [\Delta^i_j] \circ [F^j_b] = [0],$$

where

$$[\Delta^i_j] = [F^i_a] \circ d[G^a_j] = -d[F^i_a] \circ [G^a_j]$$

equivalent to Shipov (5.27) and Shipov (5.24) respectively, and

$$Possibility\ 2 : d[F^i_b] - [F^i_a] \circ [C^a_b] = [0],$$

where

$$[C^a_b] = -d[G^a_j] \circ [F^j_b] = [G^a_m] \circ d[F^m_b].$$

The matrix $[C^a_{bc} d\xi^c]$ defines the Cartan matrix of connection 1-forms. In Shipov notation,

$$[C^a_{bc} d\xi^c] \equiv [\Delta^a_{bc} d\xi^c],$$

and the two preceding equations are recognized as being related to Shipov (5.65) and (5.67). Not until the partial derivatives are required is it necessary to state how (and where) the arguments ξ^c of the connections are defined

Herein, the matrix $[\Delta^i_{jc} d\xi^c]$ will be defined as the Shipov matrix of connection.1-forms, where the matrix $[C^a_{bc} d\xi^c]$ will be defined as the Cartan matrix of connection 1-forms. The two matrices are distinct, but are (negative) similarity conjugates of one another.

$$[\Delta^i_{jc} d\xi^c] = -[F^i_a] \circ [C^a_{bc} d\xi^c] \circ [G^b_j].$$

It follows that the total exterior differentials of the basis frame can be expressed as either

$$\text{Shipov}: \quad d[F^i_b] + [\Delta^i_{jc} d\xi^c] \circ [F^j_b] = 0$$

or

$$\text{Cartan}: \quad d[F^i_b] - [F^i_a] \circ [C^a_{bc} d\xi^c] = 0.$$

The choice of plus or minus signs depends upon historical conventions.

Equivalent formulas can be given for the differentials of the Inverse matrix:

$$\text{Shipov}: \quad d[G^a_j] - [G^a_m] \circ [\Delta^m_{jc} d\xi^c] = 0$$

or

$$\text{Cartan} \;:\; d\left[G^a_j\right] + [C^a_{bc} d\xi^c] \circ \left[G^b_j\right] = 0.$$

The Vierbeins and Cartan's Equations of Structure

Given the Inverse matrix $[G^a_k]$ it is possible to define a set of 1-forms, defined as the Vierbeins, such that (compare Shipov 5.66)

$$|\sigma^a\rangle = [G^a_k] \circ |dx^k\rangle.$$

However, these Vierbein 1-forms may or may not be exact, closed, or integrable in the sense of Frobenius. From $[F^k_a] \circ |\sigma^a\rangle = |dx^k\rangle$, and the assumption, $d|dx^k\rangle = 0$, it follows that

$$[F^k_a] \circ \{|d\sigma^a\rangle + [C^a_{bc} d\xi^c] \circ |\sigma^b\rangle\} = [F^k_a] \circ |\Sigma^a\rangle = 0,$$

such that the Cartan Torsion 2-forms $|\Sigma^a\rangle$ generated by the first Cartan structural equation, must vanish:

$$|\Sigma^a\rangle = \{|d\sigma^a\rangle + [C^a_{bc} d\xi^c] \circ |\sigma^b\rangle\} = 0.$$

This result is valid whether or not the vierbiens are integrable.

The second structural equation follows by exterior differentiation of

$$[F^i_e] \circ \{[C^e_{af} d\xi^f]\hat{}[C^a_{bc} d\xi^c] + d[C^e_{bc} d\xi^c]\} = [F^i_e] \circ [\Theta^e_b] = 0,$$

such that Cartan matrix of Curvature 2-forms $[\Theta^e_b]$ must vanish.

$$[\Theta^e_b] = \{[C^e_{af} d\xi^f]\hat{}[C^a_{bc} d\xi^c] + d[C^e_{bc} d\xi^c]\} = 0.$$

The two constraints

$$[\Theta^e_b] = 0 \quad and \quad |\Sigma^a\rangle = 0,$$

on an n dimensional variety define a space of Absolute Parallelism. (In this case, an A4). The only requirement is that there exist a projective domain of support for the differentiable functions of the Frame matrix. These equations are equivalent to Shipov 5.73 to 5.76 where Shipov has defined the Cartan connection $[C^a_b]$ by the symbol $[\Delta^a_b]$. (Note that $[\Delta^a_b] \neq \left[\Delta^i_j\right]$).

Integrability

If a Basis Frame $[F^i_e(\xi^a)]$ is specified on a domain of independent variables, and it is presumed that the differentials $d\xi^a$ are well defined and exact, it is by no means clear that the induced objects $\delta x^k = F^i_e(\xi^a) d\xi^e$ are exact perfect differentials, hence integrable. In order to be integrable it is required that each of the objects δx^k satisfy the Frobenius integrability conditions (for each fixed k), $\delta x^k \hat{} d(\delta x^k) = 0$. If the conditions are not satisfied, then the δx^k are not integrable. If the conditions are satisfied, integrability may or may not require an integrating factor (for each k).

Suppose that integrability is satisfied without an integrating factor. Then a necessary condition is that

$$dF^i_c(\xi^a)\hat{}d\xi^c = F^i_a(C^a_{bc} - C^a_{cb}) d\xi^b \hat{} d\xi^c = 0.$$

This condition requires that the Cartan (right) connection be symmetric in the lower bc indices. In other words, integrability to exactness requires that the Affine torsion is zero.

However, the integrability conditions may be satisfied with the use of an integrating factor. In this situation the Cartan connection need not be symmetric in the lower bc indices. The integrable connection can generate an Affine torsion field. However the Topological Torsion, for each k, vanishes: $\delta x^k \hat{} d(\delta x^k) = 0$.

The third case for which there can exist a non-zero affine torsion field is when the Topological Torsion is not zero. Then the system is not uniquely integrable. In the integrable parametric example given above, the affine torsion field vanishes. For the diagonal Frame Field generated from the "square root" of the metric, (see below) the connection generates an Affine torsion field. The terms δx^k are not all exact, but do admit an integrating factor. All diagonal frame fields have this property.

Additional Metric constraints

If a metric $[\mathsf{g}_{ab}(\xi^e)]$ is defined on the domain of projective support (with inverse metric $[\mathsf{G}]$, then it is possible to define a Christoffel connection from the classic formula,

$$Christoffel : \Gamma^b_{ac} \Rightarrow \left\{ {b \atop ac} \right\} = \mathsf{G}^{be}\{\partial \mathsf{g}_{ce}/\partial\xi^a + \partial \mathsf{g}_{ea}/\partial\xi^c - \partial \mathsf{g}_{ac}/\partial\xi^e\}.$$

It is possible to decompose the Cartan (right) connection according to the formula,

$$C^b_{ac} = \left\{ {b \atop ac} \right\} + T^b_{ac}$$

where the T^b_{ac} can contain the antisymmetric parts. (see (5.77). In fact, for the Jacobian basis frame, the $T^b_{ac} = 0$.

Shipov on the other hand seems to decompose the "Shipov connection" Δ^i_{jk} for parallel transport on an A4 space into a "Christoffel part" and a "Ricci rotation" (5.28).

$$[\Delta^i_{jk}] = [\left\{ {i \atop jk} \right\}] + [T^i_{jk}].$$

It is hard to decide how to construct the $\left\{ {i \atop jk} \right\}$ as these must depend upon the metric on the final state. To substitute the values of the Christoffel symbols on the initial state seem senseless. The anti-symmetric part of the Shipov connection is (supposedly) included in the T^i_{jk}. Shipov defines the negative of the anti-symmetric component of Δ^i_{jk} as the "Torsion Field" (eq.5.20):

$$\Omega_{jk}^{..i} = -(\Delta^i_{jk} - \Delta^i_{kj}).$$

Note that the Shipov Torsion field defined from the Shipov connection (which is not necessarily zero) is NOT the same as the Cartan Torsion 2-forms (which are necessarily zero for a space of absolute parallelism), nor is it the same as the field generated by the anti-symmetric parts of the Cartan connection.

An example: The Diagonal Frame generated by the metric of Spherical Coordinates

Consider the map from spherical coordinates to euclidean coordinates given by the equations above and note that a Basis Frame of the form

$$[F] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & r & 0 \\ 0 & 0 & 0 & r\sin(\theta) \end{bmatrix}$$

generates the metric

$$[\mathsf{g}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\sin(\theta)^2 \end{bmatrix}$$

via the formula

$$[g_{ab}(\xi^c)] = [F^j_a] \circ [\eta_{jk}] \circ [F^k_b],$$

The diagonal basis frame generates the Cartan connection,

$$[C^a_b] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & dr/r & 0 \\ 0 & 0 & 0 & dr/r + \cot(\theta)d\phi \end{bmatrix}$$

and the Shipov connection

$$[\Delta^i_j] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -dr/r & 0 \\ 0 & 0 & 0 & -dr/r - \cot(\theta)d\phi \end{bmatrix}$$

The induced metric leads to Christoffel coefficients,

$$[\Gamma^a_b] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -rd\theta & -r\sin^2\theta d\phi \\ 0 & d\theta/r & dr/r & -\sin(\theta)\cos(\theta)d\phi \\ 0 & d\phi/r & \cot(\theta)d\phi & dr/r + \cot(\theta)d\theta \end{bmatrix}$$

and Ricci coefficients

$$[T^a_b] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & +rd\theta & +r\sin^2\theta d\phi \\ 0 & -d\theta/r & 0 & +\sin(\theta)\cos(\theta)d\phi \\ 0 & -d\phi/r & -\cot(\theta)d\phi & 0 \end{bmatrix}$$

with Shipov Torsion coefficients, defined as the anti-symmetric $\{\Delta^i_{jk} - \Delta^i_{kj}\}/2$ and they are equal to the negative of the Cartan torsion (anti-symmetric) coefficients

$$\omega^2_{21} = 1/2r = -\Omega^2_{21},$$
$$\omega^3_{31} = 1/2r = -\Omega^3_{31},$$
$$\omega^3_{32} = \cos(\theta)/2\sin(\theta) = -\Omega^3_{32}.$$

These results seem to agree with the Shipov presentation on page 217, for a diagonal basis frame, except that the computations presented by Shipov have been done using ijk indices, where herein the abc indices are used. (I am still trying to resolve the differences in notation and perspective. It would appear that Shipov has switched to the right Cartan matrix for the presentation on p 217).

Note the remarkable differences between the diagonal basis frame and the Jacobian basis frame given above, although both are for a spherical coordinate system.

Further exterior differentiation leads to other useful expressions and the Bianchi identities (see http://www22.pair.com/csdc/pdf/defects2.pdf)

Metric Constraints and Polarities

In Projective Geometries there exist two types of maps, called Collineations and Correlations. Collineations map points into points and hypersurfaces into hypersurfaces. In tensor analysis think of collineations as transformations that map contra-variant vectors (points) into contra-variant vectors, and co-vectors (hypersurfaces) into co-vectors. The Jacobian matrix of a map and its transpose are examples of collineations. Correlations map points into hypersurfaces and hypersurfaces into points.

The subset of symmetric correlations are defined as polarities. The usual concept of a metric as a symmetric matrix can be viewed as a map that takes contra-variant vectors into co-variant vectors. The metric tensor $[g_{\mu\nu}]$ is an example of a symmetric correlation, or a polarity.

**** More on projective geometries later

Appendix

Definition A variety $\{x\}$ is a space of Absolute Parallelism if the matrix of Cartan curvature 2-forms $[\Theta]$ and the vector of Cartan torsion 2-forms $|\Sigma\rangle$ defined on the variety vanish.

Theorem 1: Consider an arbitrary matrix $[F]$ of nxn functions on the variety $\{x\}$ where the variety is now restricted such that $det[F] <> 0$. Then the restricted domain is a space of Absolute Parallelism

Conjecture Shipov: The physical vacuum is a 4 dimensional space of Absolute Parallelism , further restricted to the Lorentz equivalence class of Basis Frame matrices, $[F]$, such that $[transposeF][M][F] = [M]$. The matrix $[M]$ is the Minkowski index matrix.

Theorem 2: The domain of support for an arbitrary 1-form of Action on a variety generates a space of Absolute Parallelism.

Theorem 3: There exist A4 spaces which DO NOT support a kinematic Velocity Field. (The unique limit of dx/ds = V(x,y,z,Ct) does not exist globally.)

Theorem 4: For a basis frame constructed from an integrable parametric map of maximal rank, the Affine torsion on the initial state is zero, and the Cartan connection generates the Christoffel symbols for the induced metric on the initial state.

Proof of Theorem 1:

From $det[F] <> 0$ there exists $[G]$ such that $[F][G] = [1]$.

Hence $[dF] = [F]\{-[dG][F]\} = [F][C]$, where $[C]$ is the matrix of Cartan 1-forms.

The matrix of Cartan curvature 2-forms is defined as

$$[\Theta] = \{[C]\hat{}[C\} + [dC]\}.$$

The Vierbein 1-forms are defined as

$$|\sigma\rangle = [G] \circ |dx\rangle.$$

The vector of Cartan torsion 2-forms is defined as

$$|\Sigma\rangle = \{[C]\circ|\sigma\rangle + |d\sigma\rangle\}$$

As $[ddF] = 0$, it follows that $[F]\circ\{[C]\hat{}[C\} + [dC]\} = 0;$ hence

$$[\Theta] = \{[C]\hat{}[C\} + [dC]\} = 0.$$

From

$$|dx\rangle = [F] \circ |\sigma\rangle$$

and as $|ddx\rangle = 0,$ it follows that $[F] \circ \{[C] \circ |\sigma\rangle + |d\sigma\rangle\}\} = 0;$ hence

$$|\Sigma\rangle = \{[C] \circ |\sigma\rangle + |d\sigma\rangle\} = 0.$$

QED

Proof of Theorem 2.

Given an arbitrary action 1-form, $A = A_x dx + A_y dy + .. - \phi dt.$

Construct algebraically $n-1$ associated direction fields $\mathbf{e}$ such that

$$i(\mathbf{e})A = 0.$$

example:

$$|\mathbf{e}_1\rangle = \left| \begin{array}{c} \phi \\ 0 \\ \vdots \\ A_x \end{array} \right\rangle$$

Construct the Frame

$$[F] = [\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{n}]$$

Choose $\mathbf{n}$ to be the direction field with components created by the components of the 1-form of Action.

Then (for a 4D variety)

$$det[F] = \phi^2(A_x^2 + A_y^2 + A_z^2 + \phi^2) \neq 0$$

on the domain of support of ϕ.

Hence the restricted domain is a space of Absolute Parallelism, by theorem 1.

These spaces are not necessarily Lorentzian. This method of generating a basis frame connects the Calculus of variations (based upon Action 1-forms) to spaces of absolute parallelism. Subspaces of spaces of absolute parallelism may or may not have domains of zero Cartan curvature or zero Cartan torsion.

If a space of absolute parallelism is further constrained to those matrices of Basis Frames which are symmetric (such as metrics) then the projective frames (defined as $det[F] \neq 0$) produce what are called polarities. Polarities are special projective transformations that establish duality relationships in projective geometry.

The Metric induced connection $[\Gamma]$ of 40 component functions may only be part of the Cartan connection $[C]$ of 64 component functions.

Shipov asserts that the Cartan connection on metric spaces can be composed of a Christoffel connection (metric and its derivatives) plus another component dependent algebraically upon the Cartan coefficients and the metric components.

Shipov Torsion is related to an antisymmetric combination of the Cartan Connection coefficients. (It is NOT the same as the Anti-symmetric part of the Cartan matrix of connection 1-forms, and it is not the same as Topological Torsion, for Shipov torsion can exist when the Topological Torsion is zero)

Proof of theorem 3:

Assume that

$$|dx\rangle = |V\rangle ds$$

exists globally.

Then, from the definition of the Vierbeins,

$$|\sigma\rangle = [G] \circ |dx\rangle = [G] \circ |V\rangle ds = |W\rangle ds.$$

where the W are functions.

IF true, then

$$|d\sigma\rangle = |dW\rangle\hat{}ds,$$

and

$$|\sigma\hat{}d\sigma\rangle = |Wds\hat{}dW\rangle\hat{}ds = 0.$$

Hence the Vierbeins must be integrable and of Pfaff dimension 2 at most for the kinematic Velocity field $|V\rangle$ to be defined as the limit of $|dx/ds\rangle$. The topological torsion of each Vierbein must vanish. However, there exist A4 spaces for which the Vierbiens are not of Pfaff dimension 2 at most, and the topological torsion of one or more Vierbeins is NOT zero.

Hence the assumption that $|dx\rangle = |V\rangle ds$ fails for such A4 spaces.

QED